Spacetime Diagrams II

In order to set the scale on the S' axes we need to use the spacetime interval:

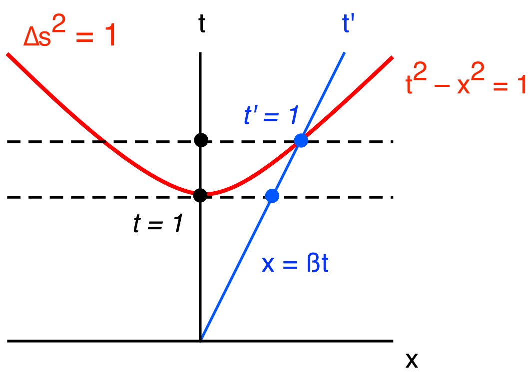
$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

The interval is important because it is invariant and all reference frames will measure the same interval between any two events. (Remember it is the 4-D spacetime version of the distance between two objects always being the same no matter the origin of the coordinates used.)

To make the math pretty simple, we will use geometric units, so that c = 1. In addition, we will continue to show only 2 dimensions – t and x. It is pretty easy to add a third dimension, but trying to graph all four dimensions makes my head hurt. So, we will keep it simple. However, that means our interval is given by

$$\Delta s^2 = \Delta t^2 - \Delta x^2$$

In the spacetime diagram below, there is a point at t = 1 and x = 0. If we imagine that is a second event, and the first event was at the origin (t = x = 0) then the interval between those events is



$$\Delta s^2 = \Delta t^2 - \Delta x^2 = 1^2 - 0^2 = 1$$

There is also a hyperbola drawn in red with the equation $t^2 - x^2 = 1$. That hyperbola would represent all the other possible events that would also have an interval of 1. That means the intersection of the red hyperbola with the blue S' time axis crosses the t' axis at t' = 1. (Because x' = 0 on the t' axis and both S and S' have to get the same interval.)

Recall that the t' axis is just the location in S of the origin of S' as S' moves to the right with a speed of ß. In S, the equation of the blue line is therefore $x = \beta t$. We can find the time of the t' = 1 event in the S frame with just a little bit of algebra as follows:

$$t^2 - x^2 = 1 \rightarrow t^2 - (\beta t)^2 = 1 \rightarrow t^2(1 - \beta^2) = 1$$

So that

$$t = \frac{1}{\sqrt{1 - \beta^2}}$$

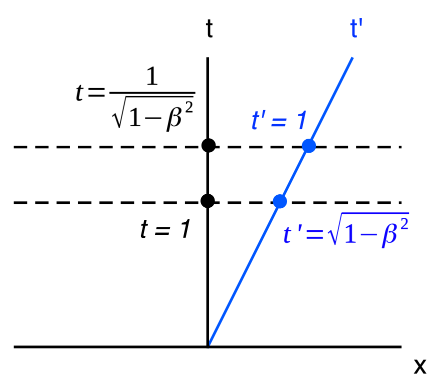
We can also find the time of the t = 1 event in the S' with a little geometry. Notice there are two similar right triangles, each using the origin as one vertex. Therefore, we can say

$$\frac{t'}{1} = \frac{1}{\frac{1}{\sqrt{1 - \beta^2}}}$$

So

$$t' = \sqrt{1 - \beta^2}$$

These are both shown in the diagram to the right.



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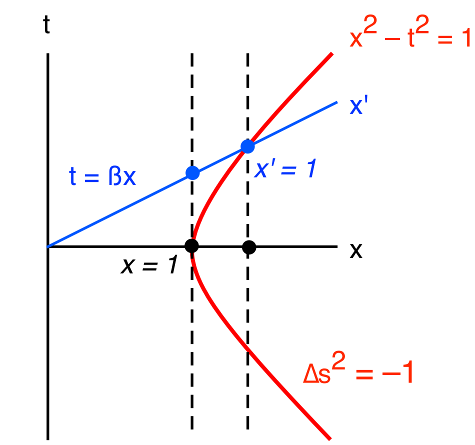
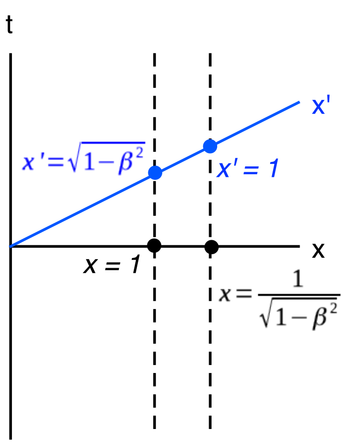
One can see our equation for time dilation right on the diagram. If S' measures the time between two events on its origin (i.e. on the t' axis), that would be the proper time. The time measured in the S frame would just be the time for that event on the t axis, so we clearly have

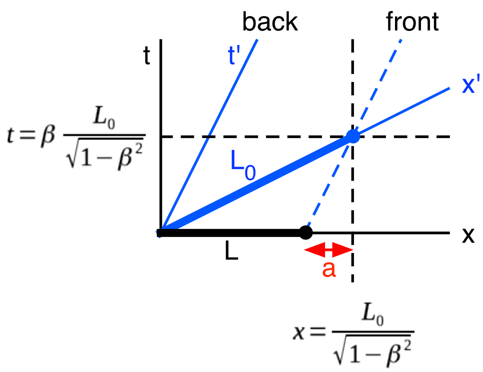
$$\boxed{t = \frac{t_0}{\sqrt{1-\beta^2}}}$$

To set the scale on the x' axis, we basically do the same thing, except we find all the events that have an interval of $\Delta s^2 = -1$, which would correspond to the hyperbola $x^2 - t^2 = 1$. The x' axis would correspond to the line $t = \beta x$. From symmetry, we will get the same answers as before, which are shown below right. (*See questions 1 and 2.*)





We can also derive the equation for length contraction from a spacetime diagram. Imagine a rod of length L_0 in its own frame (S') moving to the right with a speed of ß. The spacetime diagram for the situation is given below. For convenience, we will put the back of the rod on the origin of S and S', and analyze the worldlines of the front and back of the rod.



To determine the length, each frame of reference needs to find the coordinate of the front of the rod at the same time as the back of the rod. The bold blue line is the length in S' and the bold black line is the length in S. The S' length is the proper length, because that is the frame in which the rod is at rest.

Since $x' = L_0$ at $t'=0$, we know from the previous diagrams that the corresponding x coordinate is

$$x = \frac{L_0}{\sqrt{1-\beta^2}}$$

Since the x' axis can be given by the equation $t = \beta x$, the corresponding t coordinate is

$$t = \frac{\beta L_0}{\sqrt{1-\beta^2}}$$

Since the slope of the worldline of the front of the rod (the dashed blue line) is 1/ß, we can find the small distance *a* (in red) by

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$$\frac{1}{\beta} = \frac{\frac{\beta L_0}{\sqrt{1-\beta^2}}}{a}$$

So that

$$a = \frac{\beta^2 L_0}{\sqrt{1-\beta^2}}$$

Lastly, we can find the length L of the rod in the S frame by

$$L = \frac{L_0}{\sqrt{1-\beta^2}} - a = \frac{L_0}{\sqrt{1-\beta^2}} - \frac{\beta^2 L_0}{\sqrt{1-\beta^2}} = \frac{L_0}{\sqrt{1-\beta^2}}(1-\beta^2)$$

Which becomes our original equation for length contraction

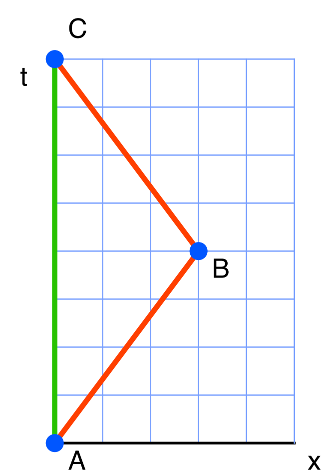
$$\boxed{L = L_0\sqrt{1-\beta^2}}$$

Problems

1. Show that the x coordinate that corresponds to x' = 1 and t' = 0 is $x = \frac{1}{\sqrt{1-\beta^2}}$.

2. Show that the t coordinate that corresponds to x' = 1 and t' = 0 is $t = \frac{\beta}{\sqrt{1-\beta^2}}$.

3. The diagram to the right shows the worldlines of two objects, one in green the other in red. Describe what the diagram shows. (No numbers needed.)



4. On the green worldline, what is the time between event A and C? (Let's say each block is 1 lyr wide and 1 yr tall.)

5. How fast is the red object moving?

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6. On the red worldline, how much time passed between A and B? (Be careful, remember the scale is different for the moving frame.)

7. On the red worldline, how much time passed between A and C?

8. What happened to the red worldline at B?

9. The two worldlines started together at A and met again at C. If the two objects were two twins, would anything weird have happened?

Answers:

3. red and green start together at A, red has a constant speed to B, turns around, and returns to it starting point with the same speed, meeting up with green again at C. green was at rest the whole time.

4. 8 years 5. ß = ¾ 6. 2.65 years 7. 5.3 years

8. It turned around, which means it changed reference frames and was accelerating for a time.

9. The twin who stayed behind aged more than the twin who went on the trip.